A BOUNDARY PROBLEM FOR A QUASILINEAR ELLIPTIC EQUATION

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A question of boundary problem decidability is often reduced to the existence problem of fixed points of some transformations in Banach spaces [1], [2].

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geqslant 2$, with a smooth boundary S. We say that a function $u(x)$ in Ω satisfies the Hölder condition (continuous according to Hölder) with the index $\alpha \in (0,1)$ if

$$\langle u\rangle_\Omega^{(\alpha)} = \sup_{x,x'\in\Omega} \frac{|u(x)-u(x')|}{|x-x'|^\alpha} < +\infty. \tag{1}$$

Denote by $C^{l+\alpha}(\overline{\Omega})$, $l \in \mathbb{N}$, the Banach space of continuous functions Ω having continuous derivatives of order $\leqslant l$ with the norm

$$|u|_\Omega^{(l+\alpha)} = \sum_{|k|=0}^{l} \sup \left|D^k u(x)\right| + \sum_{|k|=0}^{l} \sup \langle D^k u(x)\rangle_\Omega^{(\alpha)}, \tag{2}$$

where $k = (k_1, k_2, \ldots, k_n)$ is the multi-index, $k_i \geqslant 0$, $i = 1,2,\ldots,n$, and

$$D^k = \frac{\partial^{k_1+k_2+\ldots+k_n}}{\partial x_1^{k_1}\partial x_2^{k_2}\ldots\partial x_n^{k_n}}.$$

We consider the following problem:

$$L(u) \equiv \sum a_{ij}(x,u,u_x)\frac{\partial^2 u}{\partial x_i \partial x_j} + a(x,u,u_x) = 0, \quad x \in \Omega \tag{3}$$

$$M(u) \equiv b(x,u,u_x) + \sum b_i(x,u)\frac{\partial u}{\partial x_i} + b_0(x,u) = 0, \quad x \in S \tag{4}$$

with assumption that equation (3) is uniformly elliptic, i.e.

$$\sum a_{ij}(x,u,u_x)\xi_i\xi_j \geqslant \nu|\xi|^2, \ \nu = const > 0, \tag{5}$$

and

$$\sum [b_{p_i}(x,u,p) + b_i(x,u)]\cos(\mathbf{n}, x_i) \geqslant \nu_1 \max(|u|,|p|), \ \nu_1 > 0, \tag{6}$$

where $\mathbf{n}$ is an exterior normal to S.

The question of solvability for the boundary problem (3)–(4) is reduced to the fixed points problem of transformations in Banach spaces in the following way.

Let us write the boundary condition (4) as follows:

$$\left\{\sum \left[\int_0^1 \frac{\partial b(x,u,p)}{\partial p_i}\bigg|_{p=tu_x} dt + b_i(x,u)\right] u_{x_i} + u\right\}\bigg|_S = \left\{ -b(x,u,0) - b_0(x,u) + u\right\}\bigg|_S.$$

Let $v(x) \in C^{1+\alpha}(\overline{\Omega})$. Denote

$$\widehat{a}_{ij} = a_{ij}(x, v, v_x), \quad \widehat{a}(x) = a(x, v, v_x),$$

$$\widehat{b}_i(x) = \int_0^1 \frac{\partial b(x, v, p)}{\partial p_i}\Big|_{p=tv_x} dt + b_i(x, v),$$

$$\widehat{\varphi}(x) = -b(x, v, 0) - b_0(x, v) + v(x).$$

and consider the following problem:

$$\sum \widehat{a}_{ij} u_{x_i x_j} + \widehat{a}(x) = 0, \tag{7}$$

$$\left[\widehat{b}(x) u_{x_i} + u\right]\Big|_S = \widehat{\varphi}(x). \tag{8}$$

The problem (7)–(8) is a linear problem with respect to the function $u(x)$. In general we obtain a nonlinear transformation $u = \Phi(v)$. Fixed points of this transformation will be solutions of the problem (3)–(4).

Let L_0 and M_0 be differential operators of the same type as L and M respectively such that the problem (3)–(4) is uniquely solvable in $C^{2+\alpha}(\overline{\Omega})$.

We include the problem (3)–(4) into a family of problems depending on a parameter $\tau \in J = [0, 1]$:

$$L_\tau(u) \equiv \tau L(u) + (1 - \tau) L_0(u) = 0, \quad x \in \Omega, \tag{9}$$

$$M_\tau(u) \equiv \tau M(u) + (1 - \tau) M_0(u) = 0, \quad x \in S. \tag{10}$$

If $\tau = 1$ then the problem (9)–(10) becomes the problem (3)–(4).

Define the transformation A in the following way: to an element (u, v, τ), $u \in C^{2+\alpha}(\overline{\Omega})$, $v \in C^{1+\alpha}(\overline{\Omega})$, $\tau \in J$, we assign the space of pairs $\{f, u\}$, where $f \in C^{\alpha}(\Omega)$, $\varphi \in C^{1+\alpha}(S)$ with the norm $|f|_{\Omega}^{(\alpha)} + |\varphi|_S^{(1+\alpha)}$.

Suppose that S is a surface of the class $C^{2+\alpha}$, functions a_{ij}, a, b, b_i, b_0 and their partial derivatives of the first order are smooth, and their prior estimates are known.

Under these assumptions the one-valued solvability of the problem (3)–(4) in the space $C^{2+\alpha}(\overline{\Omega})$ arises from the following theorem about fixed points.

Theorem 1 *Let B_1 and B_2 be two Banach spaces, u, v, τ be elements of B_1, B_2 and J, respectively. Let A be a continuous mapping of direct product $B_1 \times B_2 \times J$ into the Banach space B_1, satisfying the following conditions:*

1) the operator $A_u(u, v, \tau)$ (the Frechet derivative) exists, the equation $A_u(u, v, \tau) = 0$ is uniquely solvable;

2) the set of solutions of the equation $A_u(u, v, \tau) = 0$ is compact in B_1;

3) for some $\tau \in [0, 1]$, there exists a unique solution u of the equation $A(u, v, \tau) = 0$.

Then for each τ the equation $A(u, v, \tau) = 0$ is one-valued solvable.

REFERENCES

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